LIVE LOAD ON AN INELASTIC HALF-PLANE

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A. M. SKOBEEV and L. M. FLITMAN
(Moscow)
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1. The plane problem of the effect of a load moving at constant velocity on the boundary of an ideal medium filling a half-space is considered.

The relationship between the pressure p and the relative volume expansion ε is assumed linear for both loading and unloading (Fig. 1). Relative to the pressure applied to the boundary it is assumed that it has a "front" moving at a velocity D exceeding the speed of sound in the medium, it decreases monotonically behind the front, and the loading profile in a frame of reference moving together with the front does not change with time.

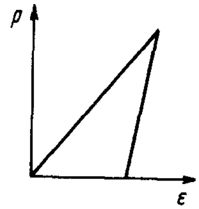

Fig. 1

The motion in the medium is assumed steady.

An abrupt dynamic effect on the soil (ground) can be modelled approximately by such a scheme. Indeed, Fig. 1 corresponds to the compressibility diagram for soil [1]. Moreover, in models used to describe the soil [2, 3] the tangential stresses are considered bounded, and it can be assumed that the stress tensor is spherical as a satisfactory approximation in such problems.

It should be added that nonuniform dynamical problems for an elastoplastic medium are quite difficult, and the soil parameters are not stable and have been studied only slightly. All this affords a foundation for the application of simple schemes, and even more so, since the model of an incompressible ideal fluid was applied to certain problems of soil dynamics [4].

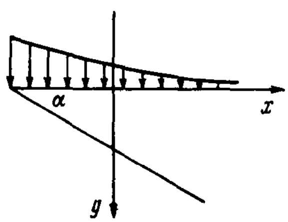

Fig. 2

The problem formulated above is useful also to clarify the limits of applicability of the approximate method, proposed in [5], for constructing the wave field in a half-space of elastoplastic material in which the motion is caused by a load moving rapidly on the boundary.

2. Let us place the origin of a rectangular coordinate system on the boundary, and let us direct the x-axis along the boundary, the y-axis into the depths of the medium (Fig. 2). Let p be the pressure, and u and v, respectively, the velocity projections on the x- and y-axes, and ρ the density.

The pressure

$$p = f(Dt + x) \tag{2.1}$$

is given at $y = 0$, where $f(\xi)$ is a known function such that

$$f(\xi) = 0 \quad (\xi < 0), \qquad f(\xi) > 0, \qquad f'(\xi) < 0 \quad (\xi > 0) \tag{2.2}$$

and $D > c$, where c is the speed of sound in the medium.

We shall assume the medium to be at rest ahead of the wave front being propagated at a velocity c corresponding to the loading branch of the p-ε diagram, the material to be loaded instantaneously on the front, and unloading to occur immediately behind the front $y = \operatorname{tg}\alpha\,(Dt + x)$.

It will later be verified that the constructed solution does not contradict these assumptions.

From the mass and momentum conservation conditions we obtain on the front

$$p = \rho c\,(-u \sin\alpha + v\cos\alpha) \qquad (y = (Dt + x)\operatorname{tg}\alpha,\ \ \sin\alpha = c/D) \tag{2.3}$$

We consider the pressure and velocity to be zero behind, far from the front

$$p = 0,\ u = 0, \quad v = 0 \quad (x = \infty) \tag{2.4}$$

In conformity with the assumptions made, we have within the angle α (Fig. 2), where the medium moves

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{p}{\rho} = 0, \qquad \frac{\partial v}{\partial t} + \frac{\partial}{\partial y}\frac{p}{\rho} = 0$$

$$\frac{\partial}{\partial t}\frac{p}{\rho} + c_1^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad \left(c_1^2 = \frac{K}{\rho}\right) \tag{2.5}$$

The coefficient K corresponds to the unloading branch of the p - ε diagram. We shall assume that

$$c < D < c_1 \tag{2.6}$$

3. Considering the motion steady, i.e. the pressure and velocity to depend only on $\xi = Dt + x$ and y, we obtain from the first equation in (2.5) and the conditions (2.4) at infinity

$$u = -p/\rho D \tag{3.1}$$

Taking account of this relationship and (2.6) and introducing the new variables

$$\xi = Dt + x, \qquad \eta = ky\,(k = \sqrt{1 - D^2/c_1^2}) \tag{3.2}$$

we reduce (2.5) to the form

$$\frac{\partial}{\partial \xi}\,p = \frac{\partial}{\partial \eta}\frac{\rho D}{k}\,v, \qquad \frac{\partial}{\partial \eta}\,p = -\frac{\partial}{\partial \eta}\frac{\rho D}{k}\,v$$

The Cauchy-Riemann equations will be satisfied if p and $\rho D/kv$ are considered the real and imaginary parts, respectively, of an analytic function $Q\,(\zeta)$

$$Q\,(\zeta) = p + i\,\frac{\rho D}{k}\,v \qquad (\zeta = \xi + i\eta)$$

The function $Q\,(\zeta)$ should be regular within an angle of magnitude β in the complex ζ-plane; hence, $\operatorname{tg}\beta = k \operatorname{tg}\alpha$. It follows from condition (2.1)

$$\operatorname{Re} Q = f\,(\xi) \qquad (\eta = 0,\ \xi \geqslant 0) \tag{3.3}$$

Condition (2.3) reduces to

$$\operatorname{Re} Q = \operatorname{tg}\beta \operatorname{Im} Q \qquad (\zeta = re^{i\beta}) \tag{3.4}$$

There results from condition (2.4) that Q should tend to zero when ζ tends to infinity while remaining within the angle. The formulated problem is solved by a well-known method [6]. The conformal mapping

$$z = \zeta^{\nu} \qquad (\nu = \pi/\beta) \tag{3.5}$$

transforms the interior of the angle into the upper half-plane.

It follows from condition (3.4) that $\operatorname{Im} ie^{i\beta}Q = 0$ on the negative part of the real axis in the z-plane.

This permits analytic continuation into the lower half-plane, and the problem reduces to finding a function which is zero at infinity, regular outside the positive part of the real axis, and satisfying the condition on it

$$Q_+ = e^{i2\beta}\,Q_- + 2f\,(t^{1/\nu})$$

Here Q_+ and Q_- are the upper and lower bounds of $Q\,(z)$, respectively. The solution of this problem, which is bounded at the origin and zero at infinity, is unique and has the form [6]

$$Q(z)=\frac{z^{1-1/\nu}}{\pi i}\int_0^\infty \frac{f(t^{1/\nu})\,dt}{t^{1-1/\nu}(t-z)}$$

Or returning to the variable $\zeta=\xi+i\eta$, we obtain

$$Q(\zeta)=-\frac{1}{i\beta\zeta}\int_0^\infty \frac{f(\tau)\,d\tau}{1-\tau^\nu/\zeta^\nu} \tag{3.6}$$

4. The constructed solution satisfies the assumption made earlier that the medium is in the unloading stage behind the wave front. Indeed, $\partial p/\partial t = D\,\partial p/\partial\xi$,therefore

$$\partial p\ /\ \partial t = D\operatorname{Re} Q'(\zeta)$$

It is easy to verify that

$$Q'(\zeta)=-\frac{1}{i\beta\zeta^2}\int_0^\infty \frac{\tau f'(\tau)\,d\tau}{1-\tau^\nu/\zeta^\nu}$$

It hence follows that on the positive part of the real axis

$$\operatorname{Re} Q' = \xi^{-1}f'(\xi) \tag{4.1}$$

For $\zeta = re^{-i\beta}$ we have

$$\operatorname{Re} Q'=\frac{\sin\beta}{\beta r^2}\int_0^\infty \frac{\tau f'(\tau)\,d\tau}{1+\tau^\nu/r^\nu} \tag{4.2}$$

The condition $f'(\xi)<0$ together with (4.1) and (4.2) and the evident relationship $Q'(\infty)=0$ yields that $\operatorname{Re} Q'<0$ on the boundary, and by virtue of the known properties of harmonic functions, is negative everywhere, which indeed proves the assertion $\partial p/\partial t<0$.

5. Let us examine the case when $c \ll D$ or

$$\beta\ll 1,\ \nu\gg 1 \tag{5.1}$$

Taking account of (2.6), we note that in this case it is necessary to require that the unloading occurs under incompressibility conditions.

To obtain the asymptotics let us rewrite $Q(\zeta)$ as

$$Q(\zeta)=-\frac{1}{i\beta\zeta}\int_0^\infty \frac{f(\tau)-f(r)-f'(r)(\tau-r)}{1-(\tau/\zeta)^\nu}\,d\tau+$$

$$+(1+i\operatorname{ctg}\beta)[f(r)-rf'(r)]+\zeta(1+i\operatorname{ctg}2\beta)f'(r)$$

$$\zeta=re^{i\varphi},\qquad 0\leqslant\varphi\leqslant\beta$$

Retaining terms not higher than second order in β in this equality, we obtain for the pressure p:

$$p=f(r)+\frac{\varphi}{\beta}\left[\frac{1}{r}\int_0^r f(\tau)\,d\tau-f(r)\right]-\beta^2F_1\left(r,\frac{\varphi}{\beta}\right) \tag{5.2}$$

$$F_1(r,\gamma)=\frac{\gamma^3}{6}\left[\frac{1}{r}\int_0^r f(\tau)\,d\tau-f(r)\right]-\frac{1}{6}\gamma(4-3\gamma)\,rf'(r)$$

Let us note that

$$\beta = kc\ /\ D + O(c^3\ /\ D^3) \tag{5.3}$$

For the vertical velocity component v we have

$$\rho cv=\frac{1}{r}\int_0^r f(\tau)\,d\tau-\beta^2F_2\left(r,\ \frac{\varphi}{\beta}\right) \tag{5.4}$$

$$F_2(r, \gamma) = \frac{\gamma^2}{2r}\int_0^r f(\tau)\,d\tau + \frac{1}{3}(1-\gamma^2) f(r) + \frac{1}{3}\left(1 - 3\gamma + \frac{3}{2}\gamma^2\right) r f'(r) \qquad \text{(cont.)}$$

It is seen from (5.2) and (5.4) that the corrections to the final quantities are of second order in β.

The finite part of the pressure is a linear function of the angle, or equivalently, the depth, while the finite part of the vertical velocity is not dependent on the angle.

It follows from (3.1) that the horizontal velocity is a quantity on the order of β

$$u = -\beta p / k\rho c$$

For large values of r we have

$$p = f(r) + \frac{\varphi}{\beta}\left(1 - \frac{\varphi^2}{6}\right)\frac{P}{r}, \qquad \rho c v = \left(1 - \frac{\varphi^2}{2}\right)\frac{P}{r} \quad \left(P = \int_0^\infty f(\tau)\,d\tau\right)$$

It is assumed that the integral converges. It follows from these formulas that the pressure on the front always decreases as r^{-1}.

Let us compare the finite part of the pressure and velocity with the solution of the problem of motion in a semi-infinite rod of material described by a linear diagram for loading and incompressible for unloading. Motion is caused by the pressure applied to an endface at time $t = 0$ and changing according to the law $p = f(Dt)$.

In this case we have for the velocity

$$v = \frac{1}{c\rho Dt}\int_0^{Dt} f(\tau)\,d\tau$$

and the pressure on the front is given by the formula $p = c\rho v$.

Taking into account that $Dt = r$, we note that the main terms of (5.2) and (5.4) agree with the formulas presented. This can be a foundation for utilizing one-dimensional problems to construct approximate solutions, as has been mentioned in [5].

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